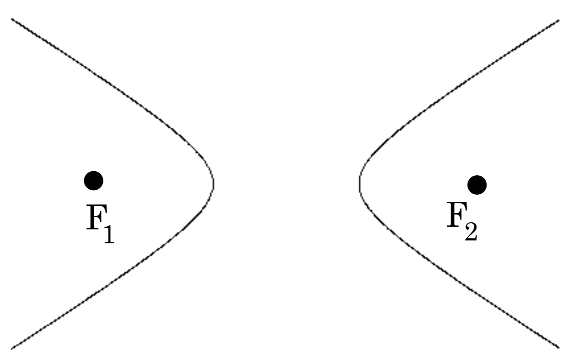
CONIC SECTIONS – THE HYPERBOLA

MOST of this is in Hyperbolas.docx.

❑ Introduction

❑ Definition of a Hyperbola

Let F_1 and F_2 be two distinct points in the plane and let c be a positive constant. Then a ***hyperbola*** (hy-PER-bo-la) is the set of points in the plane satisfying the following condition: A point is on the hyperbola if the difference of its distances to F_1 and F_2 is c, the given constant. Each of the points F_1 and F_2 is called a **focus**.



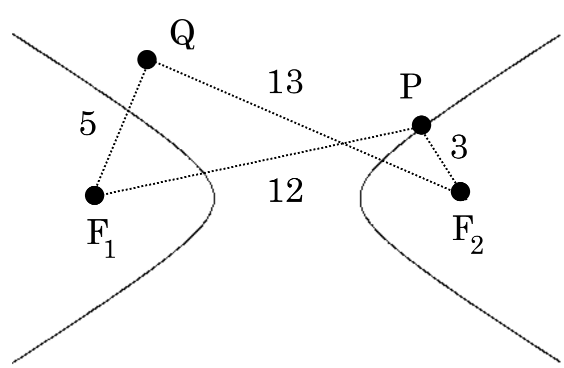
Apollonius (Greek: 262 BC – 190 BC) gave the hyperbola its present name. See how the graph of the hyperbola seems to "fly away" in all directions? Look up the word "hyperbol<u>e</u>" in a dictionary and see if its meaning has any connection to the graph of a hyperbola.

EXAMPLE 1: **In the hyperbola below assume that the constant mentioned in the definition of the hyperbola is 9. Explain why point P is on the hyperbola and why point Q is not on the hyperbola.**

Solution: Again we use the notation $d(\text{A}, \text{B})$ to denote the distance between the points A and B. By definition, a point is on the hyperbola if the difference of its distances to the two foci is constant — in this example, **9**.

Point P: $d(\text{P}, \text{F}_1) = 12$ and $d(\text{P}, \text{F}_2) = 3$, and so the difference of the distances is 9. Therefore, P is on the hyperbola.

Point Q: $d(\text{Q}, \text{F}_1) = 5$ and $d(\text{Q}, \text{F}_2) = 13$, which means that the difference of the distances is 8, not 9. We conclude that Q is not on the hyperbola.

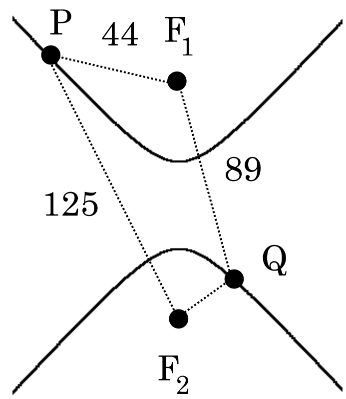


Homework

1. Sketch a hyperbola with the foci on the same vertical line.

2. Let F_1 and F_2 represent the foci of a hyperbola. Also suppose that P is a point on the hyperbola. Assume further that the distance between P and F_1 is 20 and the distance between P and F_2 is 12. Now let Q be a point on the hyperbola such that the distance from Q to F_1 is 30. What is the distance between Q and F_2?
   a. 1 b. 2 c. 22 d. 15 e. 14

3. In the hyperbola below, find the distance between Q and F_2.



4. Is it possible for a hyperbola to be a function? (We assume just the two kinds of hyperbolas discussed in this chapter — there are others.)

❑ The Main Example

Graph the hyperbola $x^2 - y^2 = 9$.

First, let's find the **domain**. We solve for y, and then analyze all the legal values of x:

$x^2 - y^2 = 9$	(the original equation)
$\Rightarrow -y^2 = -x^2 + 9$	(subtract x from each side)
$\Rightarrow y^2 = x^2 - 9$	(multiply each side by -1)
$\Rightarrow y = \pm\sqrt{x^2 - 9}$	(take both square roots)

Because of the square root, y is a real number only when the radicand, $x^2 - 9$, is greater than or equal to zero: $x^2 - 9 \geq 0$. You can solve this inequality using the Boundary Point Method. When you do, you will conclude that the domain of the graph is $\mathbf{(-\infty, -3] \cup [3, \infty)}$. By the way, can you explain why the hyperbola is <u>not</u> a function?

Second, we'll check out **intercepts**. Since 0 is not in the domain, there cannot possibly be any y-intercepts. Indeed, letting $x = 0$ gives $0^2 - y^2 = 9 \Rightarrow y^2 = -9 \Rightarrow y = \pm\sqrt{-9}$, not members of $\mathbb{R}$. Thus, there are no y-intercepts. As for x-intercepts, let $y = 0$; this gives $x^2 - 0^2 = 9 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$. Therefore, there are two x-intercepts: **(3, 0)** and **(–3, 0)**.

Third, let's tackle **symmetry**. Replace x with $-x$: $(-x)^2 - y^2 = 9$, which is equivalent to $x^2 - y^2 = 9$, the original equation. Thus, the graph has y-axis symmetry. Similarly, you can show that the graph has x-axis symmetry. In fact, the graph of the hyperbola also has origin symmetry.

Fourth, we analyze **asymptotes**. Since there are no fractions in the hyperbola formula, the graph appears to have no vertical asymptotes. To find other possible asymptotes, we'll let $x \to \infty$ and see if y approaches anything interesting.

Let's construct a little x-y chart, and then choose really big values for x. Use your calculator to confirm these results.

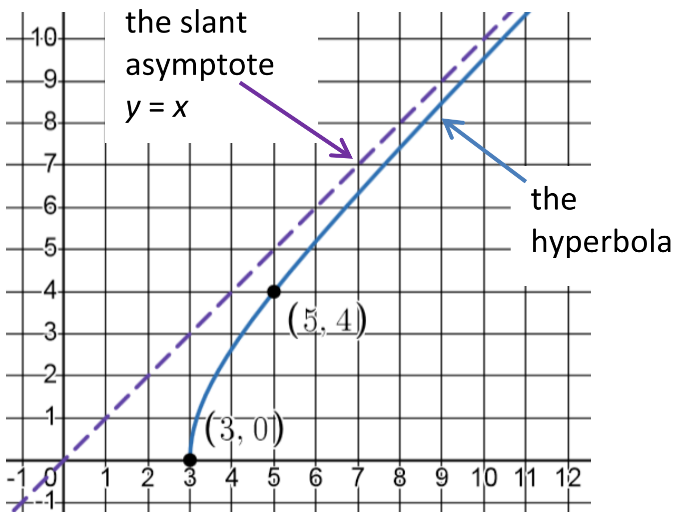
x	y
5	$\pm$ 4
10	$\pm$ 9.5394
100	$\pm$ 99.5499
1,000	$\pm$ 999.9955
10,000	$\pm$ 9,999.9996

What's going on here? Let's look just at the positive y-values. On the surface, it appears that as x grows larger and larger, the y-values grow larger and larger. But look more carefully; the y-value is actually approaching the x-value! That is, as $\boldsymbol{x \to \infty, y \to x}$. We can write this limit as

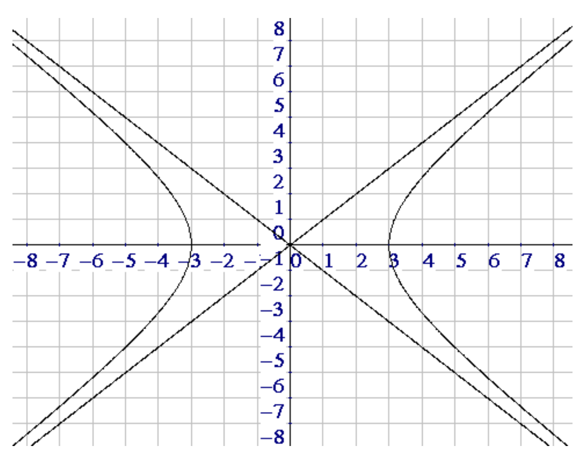
$$\lim_{x \to \infty} \sqrt{x^2 - 9} \;=\; x$$

This means that as x grows larger and larger, the curve is approaching the line $y = x$ (a 45° line passing through the origin). Since this line is neither vertical nor horizontal, we need a new adjective. We call this kind of asymptote a ***slant***, or *oblique*, asymptote.

The completed analysis gives a picture of the hyperbola in the first quadrant. It has an x-intercept at (3, 0), and a slant asymptote of $y = x$. Also, from the chart above, the hyperbola passes through the point (5, 4).



Since the hyperbola has y-axis symmetry, another piece of the hyperbola lies in the second quadrant. But the pieces of the hyperbola in Quadrants I and II have symmetric pieces in Quadrants III and IV, due to the x-axis symmetry of the hyperbola. We're now ready to graph the



entire hyperbola. Note that the lines $y = x$ and $y = -x$ are the slant asymptotes of the hyperbola, and are graphed just for reference — they are <u>not</u> part of the hyperbola. Finally, the graph should convince you of the fact deduced earlier that the domain is $(-\infty, -3] \cup [3, \infty)$. In addition, the graph shows us that the **range** of the hyperbola is $\mathbb{R}$.

Homework

5. Regarding the main example above, $x^2 - y^2 = 9$, explain, using the formula, why it isn't a line, a parabola, a circle, or an ellipse.

6. Find the slant asymptotes for each hyperbola:

   a. $x^2 - y^2 = 25$ b. $x^2 - 9y^2 = 9$

   c. $36x^2 - y^2 = 36$ d. $9x^2 - 4y^2 = 36$

7. Find the domain and range of each hyperbola in the previous problem.

8. Perform a complete analysis (including a graph) on the hyperbola $x^2 - y^2 = 49$.

❑ Another Example

Graph the hyperbola $\boldsymbol{y^2 - 4x^2 = 25}$.

First we solve for y — this will allow us to find the domain and the slant asymptotes.

$$y^2 - 4x^2 = 25 \Rightarrow y^2 = 25 + 4x^2 \Rightarrow y = \pm\sqrt{25 + 4x^2}$$

Look at the radicand. No matter what kind of number x is, its square x^2 is non-negative, which implies that $4x^2$ is non-negative, which implies that $25 + 4x^2$ is <u>at least</u> 25. So surely the radicand can <u>never</u> be negative. Thus, the **domain** of the hyperbola is $\mathbb{R}$.

Now it's time for the **intercepts**. Let $x = 0$; then $y^2 = 25$, or $y = \pm 5$. So we have a couple of y-intercepts: (0, 5) and (0, –5). Now let $y = 0$; then $-4x^2 = 25 \Rightarrow x^2 = -\frac{25}{4} \Rightarrow$ no real solution for x. Therefore, there are no x-intercepts.

I'll leave it to you to verify that the hyperbola possesses all three types of **symmetry**.

To find the slant **asymptotes**, we refer to the form of the equation calculated above,

$$y = \pm\sqrt{25 + 4x^2}$$

Use your calculator to confirm the following ordered pairs (using just the positive square root.):

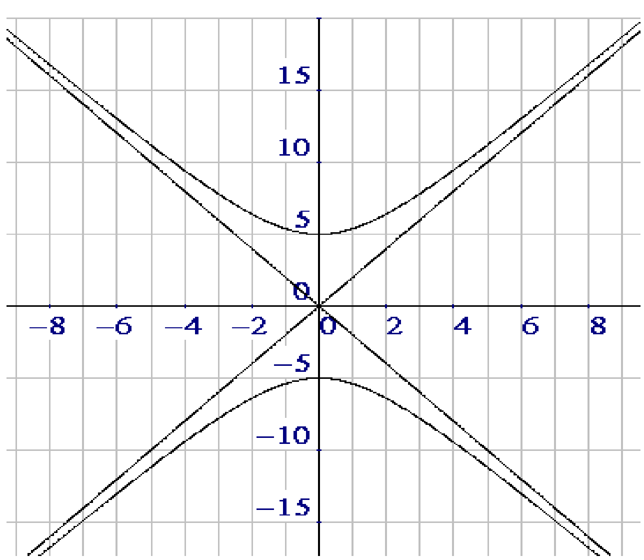
(10, 20.616) (100, 200.062) (1000, 2000.006)

As x grows larger and larger, the y-values appear to be growing closer and closer to whatever <u>twice x</u> is. That is, as $x \to \infty$, $y \to 2x$. We write

$$\lim_{x \to \infty} \sqrt{25 + 4x^2} = 2x.$$

Thus, one of the slant asymptotes is the line $y = 2x$.

The graph goes through the intercept (0, 5) and then approaches the slant asymptote $y = 2x$. The x-axis and y-axis symmetries then give us the rest of the hyperbola in the other three quadrants. As before, the asymptotes are graphed for reference.



Last, we discuss the **range** of the hyperbola. The y-values appear to be numbers that are either less than or equal to -5, or greater than or equal to 5. That is, the range of the hyperbola is $\{y \in \mathbb{R} \mid y \le -5 \text{ OR } y \ge 5\}$, which is $\mathbf{(-\infty, -5] \cup [5, \infty)}$.

Homework

9. Find the slant asymptotes for each hyperbola:

   a. $y^2 - x^2 = 144$ b. $y^2 - 16x^2 = 16$

   c. $9y^2 - x^2 = 9$ d. $25y^2 - 9x^2 = 225$

10. Find the domain and range of each hyperbola in the previous problem.

Perform a complete analysis (including a graph) on each hyperbola:

11. $y^2 - x^2 = 25$ 12. $y^2 - 9x^2 = 16$

13. $16y^2 - 9x^2 = 144$ 14. $\dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$

15. Classify each of the following as a line, parabola, circle, ellipse, or hyperbola:

   a. $7x^2 - \pi x + 2y + 9 = 0$ b. $83x + 99y = 34$

   c. $7x^2 + 2y^2 - x = 10$ d. $y^2 + 7y - x^2 - 2x - 100 = 0$

   e. $7x + 99 = 400$ f. $32 - 9y + \pi = 0$

   g. $\sqrt{2}\,x^2 + \sqrt{2}\,y^2 - \sqrt{e}\,x + \sqrt[3]{\pi}\,y - 10^6 = 0$

16.** Find the slant asympotes for $25x^2 - 9y^2 + 50x + 90y - 425 = 0$.

Review Problems

17. State the geometric definition of the hyperbola.

18. Classify as parabola, circle, ellipse, hyperbola, or none of them:

a. $x^2 + y^2 = -1$ b. $y^2 - x^2 = 5$ c. $2x^2 + 4y^2 = 7$

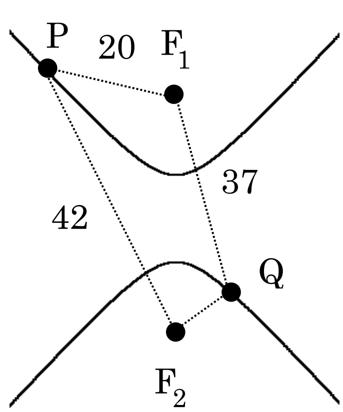
d. $x^2 - y = 3$ e. $2x^2 - \pi y^2 = \sqrt{2}$ f. $3y^2 = x^2 + 5$

19. How many foci does a hyperbola have?

20. Let F_1 and F_2 represent the foci of a hyperbola. Also suppose that P is a point on the hyperbola. Assume further that the distance between P and F_1 is 300 and the distance between P and F_2 is 160. Now let Q be a point on the hyperbola such that the distance from Q to F_1 is 110. What is the distance between Q and F_2?

a. 160 b. 180 c. 540 d. 250 e. 360

21. In the hyperbola below, find the distance between Q and F_2.



22. a. Find the intercepts of $3x^2 - 2y^2 = 2$.

  b. Find the intercepts of $10y^2 - 30x^2 = 3$.

  c. Find the domain of $x^2 - 14y^2 = 144$.

  d. Find the range of $9x^2 - 36y^2 = 41$.

  e. Find the asymptotes of $49x^2 - y^2 = 10$.

  f. Find the asymptotes of $36y^2 - 4x^2 = 1$.

  g. Find all symmetries of $4x^2 - 9y^2 = 36$.

23. Find the intercepts, domain, range, asymptotes, and symmetry of the hyperbola $x^2 - y^2 = 4$. Graph the hyperbola.

24. Find the asymptotes and the range of $y^2 - 9x^2 = 16$. Sketch the graph.

25. True/False:

  a. A hyperbola is the set of points in the plane such that the difference of the distances to two fixed points is a constant.

  b. Some of the hyperbolas in this chapter are functions.

  c. The domain of the hyperbola is $x^2 - y^2 = 9$ is $[-3, 3]$.

  d. Hyperbolas in this chapter always possess all three symmetries.

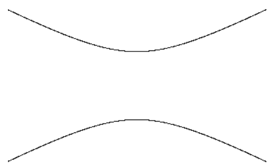
  e. The slant asymptotes for $16x^2 - 9y^2 = 1$ are $y = \pm\frac{4}{3}x$.

  f. There are no slant asymptotes for $25x^2 + 49y^2 = 1$.

  g. The range of a hyperbola may consist of the union of two intervals.

  h. Some of the hyperbolas in this chapter pass through the origin.

Solutions

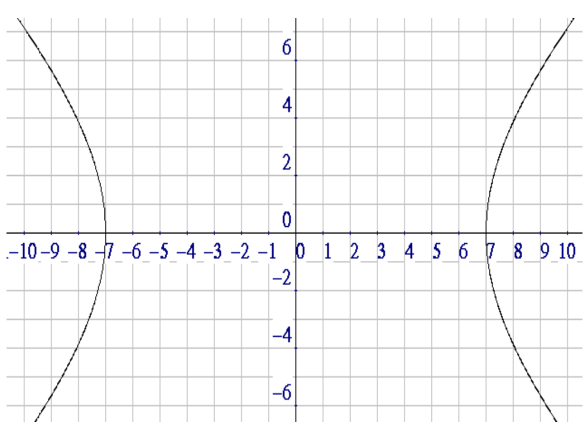
1. 

2. c. **3.** 8 **4.** No

5. Any letter squared $\Rightarrow$ it's not a line
Both letters squared $\Rightarrow$ not a parabola
Different coefficients on the squares $\Rightarrow$ not a circle
Opposite signs on the coefficients on the squares $\Rightarrow$ not an ellipse

6. a. $y = \pm x$ b. $y = \pm\frac{1}{3}x$ c. $y = \pm 6x$ d. $y = \pm\frac{3}{2}x$

7. a. $(-\infty, -5] \cup [5, \infty)$; $\mathbb{R}$ b. $(-\infty, -3] \cup [3, \infty)$; $\mathbb{R}$
c. $(-\infty, -1] \cup [1, \infty)$; $\mathbb{R}$ d. $(-\infty, -2] \cup [2, \infty)$; $\mathbb{R}$

8. 

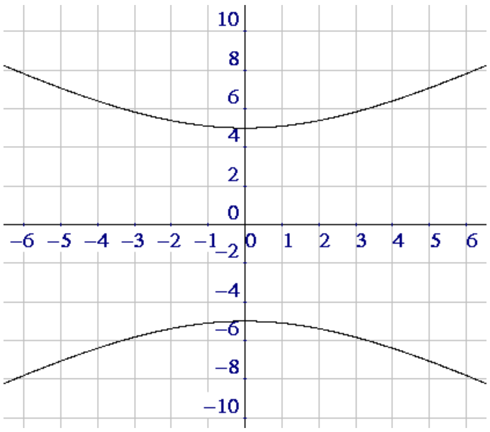
Domain $= (-\infty, -7] \cup [7, \infty)$
Range $= \mathbb{R}$ Not a function
No y-intercepts
x-intercepts: $(-7, 0)$ and $(7, 0)$
Symmetry: all three
$\lim\limits_{x \to \infty} \sqrt{x^2 - 49} = x$
Slant asymptotes: $y = x$ and $y = -x$

9. a. $y = \pm x$ b. $y = \pm 4x$ c. $y = \pm\frac{1}{3}x$ d. $y = \pm\frac{3}{5}x$

10. a. $\mathbb{R}$; $(-\infty, -12] \cup [12, \infty)$ b. $\mathbb{R}$; $(-\infty, -4] \cup [4, \infty)$

c. $\mathbb{R}$; $(-\infty, -1] \cup [1, \infty)$ d. $\mathbb{R}$; $(-\infty, -3] \cup [3, \infty)$

11.

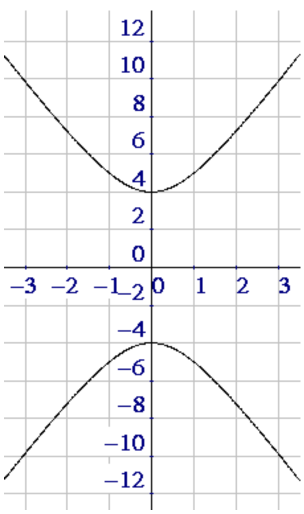


Domain = $\mathbb{R}$ Range = $(-\infty\ -5] \cup [5, \infty)$ Not a function

No x-intercept; y-intercepts: (0, 5) and (0, –5)

Symmetry: all three Asymptotes: $y = x$ and $y = -x$

12.



Domain = $\mathbb{R}$

Range = $(-\infty, -4] \cup [4, \infty)$

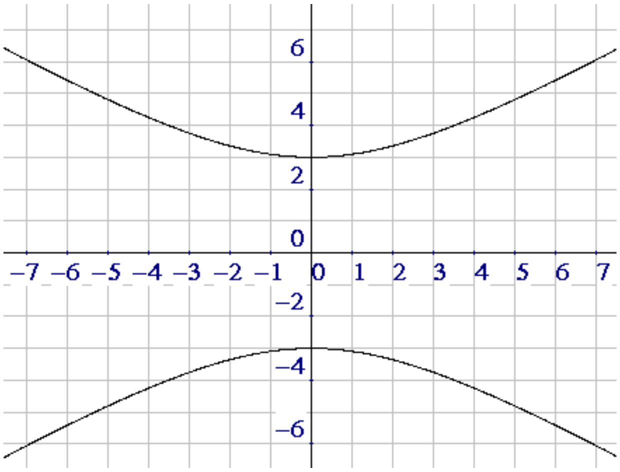
Not a function

No x-intercepts; y-intercepts: (0, 4) and (0, –4)

Symmetry: all three

Asymptotes: $y = 3x$ and $y = -3x$

13.



Domain = $\mathbb{R}$

Range = $(-\infty\ -3] \cup [3, \infty)$

Not a function

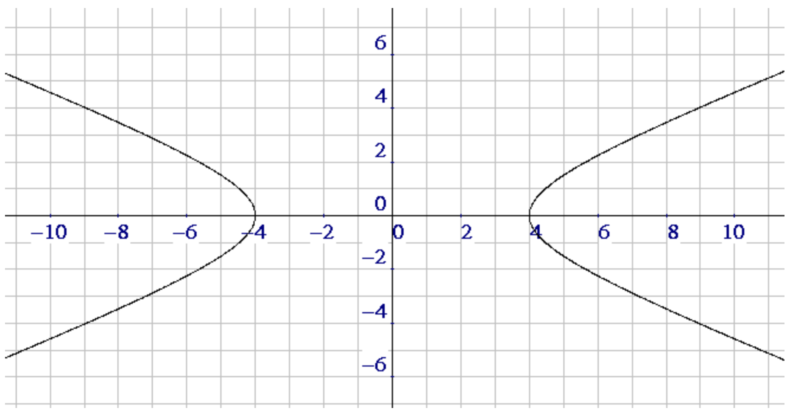
No x-intercepts

y-intercepts: (0, 3) and (0, –3)

Symmetry: all three

Asymptotes: $y = \pm\frac{3}{4}x$

14.



Domain = $(-\infty, -4] \cup [4, \infty)$ Range = $\mathbb{R}$ Not a function

Intercepts: (–4, 0) and (4, 0) Symmetry: all three

Slant asymptotes: $y = \pm\frac{1}{2}x$

15. a. parabola b. line c. ellipse d. hyperbola

e. (vertical) line f. (horizontal) line g. circle

16. $y = \frac{5}{3}x + \frac{20}{3}$ $y = -\frac{5}{3}x + \frac{10}{3}$

17. A hyperbola is the set of points in the plane such that the difference of the distances to two fixed points is a constant.

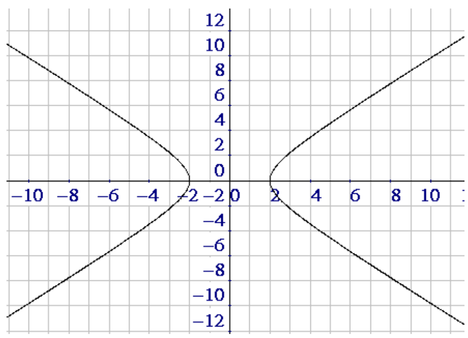
18. a. none b. hyperbola c. ellipse
d. parabola e. hyperbola f. hyperbola

19. 2 **20.** d. **21.** 15

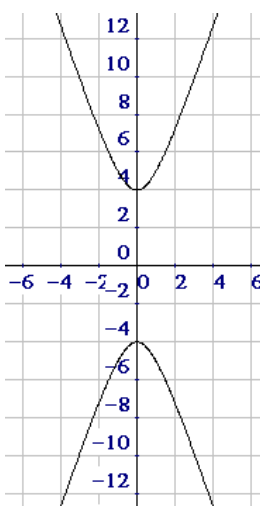
22. a. $\left(\pm\sqrt{\frac{2}{3}},\ 0\right)$ b. $\left(0,\ \pm\sqrt{\frac{3}{10}}\right)$ c. $(-\infty, -12] \cup [12, \infty)$ d. $\mathbb{R}$

e. $y = \pm 7x$ f. $y = \pm\frac{1}{3}x$ g. x-axis, y-axis, and origin

23. $(\pm 2, 0)$ Dom = $(-\infty, -2] \cup [2, \infty)$ Rg = $\mathbb{R}$ asym: $y = \pm x$
all 3 symmetries



24. $y = \pm\, 3x$; $(-\infty, -4] \cup [4, \infty)$



"Effort only fully releases its reward after a person refuses to quit."

Napoleon Hill

25. a. T b. F c. F d. T e. T f. T g. T h. F